Math 299 | **Lecture 16: Sections 5.4 - 5.5**
Existence Proofs.

Existence Proofs

Our goal in this section is to prove a statement of the form

There **exists** x for which $P(x)$. (That is, $\exists x,\ P(x)$).

I. A constructive proof of existence: The proof is to display a specific value $x = a$ in a given set and verify that $P(a)$ is true.

   EX: Prove that, for every natural number x, there exists a natural number y such that $2x - y = -1$.

II. A nonconstructive proof of existence: Use theorems which imply the existence of an x such that $P(x)$ is true without indicating how to explicitly produce such x.

   The *Intermediate Value Theorem* and the *Mean Value Theorem* are examples of existence theorems that can be used in this manner.
   EX: Prove that there exists a real number x in $[-1, 1]$ such that $2x^3 + 1 = 0$.

Examples: A constructive proof of existence

1. Prove that there are pairs of irrational numbers x and y such that x^y is rational.

2. Prove that there exists an integer x such that $\frac{8x+2}{3x-1} = 2$.

3. There exist distinct perfect squares x, y, and z such that $x + y = z$.

4. Prove that for $\varepsilon = 1$, there exists a positive real number δ such that
$|x - 2| < \delta \implies |(2x + 3) - 7| < \varepsilon$.
(We will revisit the formal definition of a limit of a function in Chapter 12.)

5. There is a *prime* number p such that $p+2$ and $p+6$ are also prime numbers.

6. There exists an even integer n that can be written in **two different** ways as a sum of two distinct primes.

Examples: A nonconstructive proof of existence

- **The Intermediate Value Theorem**: If f is a function that is continuous on the closed interval $[a,b]$ and k is a number between $f(a)$ and $f(b)$, then there **exists** a number $c \in (a,b)$ such that $f(c) = k$.

- **The Mean Value Theorem**: If a function f is continuous on the closed interval $[a,b]$, and differentiable on the open interval (a,b), then there **exists** a point $c \in (a,b)$ such that $f^{`}(c) = \frac{f(b)-f(a)}{b-a}$.

Examples

1. There exists a solution for the equation $x^3 + 3x - 2 = 0$ in the interval $(0,1)$.

2. Let $f(x) = 4x^5 - x + 2$. Prove that there exists a $c \in (0, 1)$ such that $f^{`}(c) = 3$. Note that $f(0) = 2$ and $f(1) = 5$.

Unique Existence. Examples.

1. An equation $x^5 + 4x - 1 = 0$ has exactly one solution.

2. Prove that, for every x, there exists a unique $y \in \mathbb{R}$ such that $2x + 1 = 2y - 1$.

   (1) Prove the existence y to $2x + 1 = 2y - 1$.

   (2) Prove the uniqueness by contradiction.

Disproving Existence Statements

$$\sim (\exists x \in S, P(x)) \equiv \forall x \in S, \sim P(x)$$

If the statement, "$\exists x \in S, P(x)$", is false, every $x \in S$ satisfies "$\sim P(x)$".

Examples: Disprove the statements

1. There is a real number x for which $x^4 - 6x^2 + 2 < -7$.

2. There exist odd integers a and b such that $4|(3a^2 + 7b^2)$. (Textbook).